

General Disclaimer

One or more of the Following Statements may affect this Document

- This document has been reproduced from the best copy furnished by the organizational source. It is being released in the interest of making available as much information as possible.
- This document may contain data, which exceeds the sheet parameters. It was furnished in this condition by the organizational source and is the best copy available.
- This document may contain tone-on-tone or color graphs, charts and/or pictures, which have been reproduced in black and white.
- This document is paginated as submitted by the original source.
- Portions of this document are not fully legible due to the historical nature of some of the material. However, it is the best reproduction available from the original submission.

BOCHNER'S THEOREM IN INFINITE DIMENSIONS

by

P. L. Falb¹

and

U. Haussmann²

Center for Dynamical Systems
Division of Applied Mathematics
Brown University
Providence, R. I.

N69-37025

FACILITY FORM 602

(ACCESSION NUMBER)	(THRU)
27	1
(PAGES)	(CODE)
CR-105750	19
(NASA CR OR TMX OR AD NUMBER)	(CATEGORY)



¹This research was supported in part by NSF under Grant No. GK-2788.

²This research was supported in part by National Aeronautics and Space Admin. under Grant No. NGL 40-002-015 and by the National Research Council of Canada.

BOCHNER'S THEOREM IN INFINITE DIMENSIONS

1. Introduction

Let G be a locally compact abelian group. A well-known theorem of Bochner ([1], [2]) states that a mapping ψ of G into \mathbb{C} is positive definite and continuous if and only if there is a unique nonnegative finite regular Borel measure m_ψ on \hat{G} (the dual group of G) such that $\psi(g) = \int_{\hat{G}} \gamma(g) dm_\psi$ where (γ, g) denotes the action of the character γ on g . An alternate version of the theorem ([3]) states that if A is a semi-simple, self-adjoint, commutative Banach algebra and ψ is a linear functional on A , then ψ is positive and extendable if and only if there is a finite positive Baire measure ν_ψ on \mathcal{M} (the maximal ideal space of A) such that $\psi(\alpha) = \int_{\mathcal{M}} \hat{\alpha}(M) d\nu_\psi$ where $\hat{\alpha}$ is the Gelfand transform of $\alpha \in A$. Here we shall extend these theorems to mappings taking values in a Banach space X . Our results generalize the extension of Bochner's theorem made in [4].

We shall, in fact, first prove that if A is a self-adjoint, commutative Banach algebra and ψ is a linear map of A into the Banach space X , then ψ is positive⁺ and "almost" extendable if and only if there is a weak-*⁻regular, finite, positive set function ν_ψ^{**} mapping $\Sigma(\mathcal{M})$ (the Borel field of \mathcal{M}) into X^{**} such that $\psi(\alpha) = \int_{\mathcal{M}} \hat{\alpha}(M) d\nu_\psi^{**}$ (where $\psi(\alpha)$ is viewed as an element of X^{**}). We next show that if the mapping $\hat{\psi}$ of \hat{A} into X given by $\hat{\psi}(\hat{\alpha}) = \psi(\alpha)$ is weakly compact⁺⁺, then ν_ψ^{**} can be viewed as a weakly regular positive vector measure ν_ψ .

⁺Positivity is with respect to a suitable cone in X .

⁺⁺This means that $\hat{\psi}$ maps bounded sets in A into weakly compact sets in X .

mapping $\Sigma(\mathcal{M})$ into X and, conversely, if $\psi(\alpha) = \int_{\mathcal{M}} \hat{\alpha}(M) d\nu_{\psi}$ where ν_{ψ} is a weakly regular positive vector measure on $\Sigma(\mathcal{M})$ to X , then ψ is positive and "almost" extendable and $\hat{\psi}$ is weakly compact. In the case where $A = L_1(G, C)$, these results lead to a representation of ψ by an element p_{ψ} of $L_{\infty}(G, X)$ i.e. $\psi(\alpha) = \int_G \alpha(g) p_{\psi}(g) d\mu$ where μ is the Haar measure on G . We then develop an extended Bochner's theorem for maps p in $L_{\infty}(G, X)$. Finally, we use some particular Banach spaces to illustrate the theory.

The general results obtained here are combined with the transform theory on $L_1(G, X)$ to develop an inversion theorem and a Plancherel theorem in [5]. These theorems are also applied to the solution of convolution equations in Hilbert spaces in [5]. The convolution equations arise in the study of problems relating to the stability and control of systems described by parabolic partial differential equations.

2. Positive Functions

Let X be a Banach space and let X^* and X^{**} be the dual spaces of X and X^* , respectively. If ϕ is an element of X^* , then the operation of ϕ on x is denoted by (x, ϕ) . The notion of positivity that we use is based on a cone of "positive" elements contained in X . We assume that the cone is defined by a family of elements of X^* . More precisely, we have

DEFINITION 2.1. Let Φ be a subset of X^* . The subset K_{Φ} (or simply K when Φ is fixed by the context) of X given by

$$(2.2) \quad K_{\Phi} = \{x \in X: (x, \phi) \geq 0 \text{ for all } \phi \text{ in } \Phi\}$$

is called the cone determined by Φ .

Now let A be a Banach algebra with an involution given by $\alpha \rightarrow \alpha^*$, $\alpha \in A$, and let ψ be a linear mapping of A into X . We then have

DEFINITION 2.3. The mapping ψ is positive with respect to the cone K_Φ (or Φ -positive) if $\psi(\alpha\alpha^*) \in K_\Phi$ for all α in A .

We observe that ψ is Φ -positive if and only if the mappings $(\psi(\cdot), \varphi)$ of A into C are positive functionals for all φ in Φ . Note also that if ψ is Φ -positive, then, for any φ in Φ , the functional $B_\varphi(\alpha, \beta)$ given by

$$(2.4) \quad B_\varphi(\alpha, \beta) = (\psi(\alpha\beta^*), \varphi)$$

is a symmetric bilinear form satisfying the Cauchy inequality

$$(2.5) \quad |B_\varphi(\alpha, \beta)|^2 \leq B_\varphi(\alpha, \alpha)B_\varphi(\beta, \beta)$$

for α, β in A .

DEFINITION 2.6. The mapping ψ is symmetric with respect to Φ (or simply symmetric) if $(\psi(\alpha), \varphi) = \overline{(\psi(\alpha^*), \varphi)}$ for all φ in Φ and α in A .

If A has a unit e , then every Φ -positive mapping is symmetric since $(\psi(\alpha), \varphi) = (\psi(\alpha e), \varphi) = B_\varphi(\alpha, e) = \overline{B_\varphi(e, \alpha)} = \overline{(\psi(\alpha^*), \varphi)}$ for all φ . If A does not have a unit, then A can be imbedded in an algebra $\tilde{A} = A \oplus C$ with a unit in a natural way. Letting e be the unit in \tilde{A} , we can extend

ψ to a linear mapping $\tilde{\psi}_{x_0}$ of \tilde{A} into X by setting $\tilde{\psi}_{x_0}(\alpha + ce) = \psi(\alpha) + cx_0$ for a given x_0 in X . Clearly ψ is symmetric if and only if $\tilde{\psi}_{x_0}$ is. We now have

DEFINITION 2.7. A Φ -positive mapping ψ is almost extendable if (i) ψ is symmetric, (ii) ψ is continuous, and (iii) $|(\psi(\alpha), \varphi)|^2 \leq d \|\psi\| \|\varphi\| (\psi(\alpha\alpha^*), \varphi)$ for all φ in Φ and α in A where d is a constant with $d \geq 1$.

DEFINITION 2.8. A Φ -positive mapping ψ is extendable if ψ is symmetric and if there is an x_0 in X such that $|(\psi(\alpha), \varphi)|^2 \leq (x_0, \varphi)(\psi(\alpha\alpha^*), \varphi)$ for all φ in Φ and α in A .

If A has a unit e , then any Φ -positive mapping is extendable.

If A does not have a unit, then we have

PROPOSITION 2.9. A Φ -positive mapping ψ is extendable if and only if there is an extension $\tilde{\psi}$ of ψ to \tilde{A} which is Φ -positive.

Proof: If $\tilde{\psi}$ is a Φ -positive extension of ψ and e is the unit in \tilde{A} , then, letting $x_0 = \tilde{\psi}(e)$, we deduce immediately that $|(\psi(\alpha), \varphi)|^2 = |(\tilde{\psi}(\alpha), \varphi)|^2 \leq (x_0, \varphi)(\tilde{\psi}(\alpha\alpha^*), \varphi) = (x_0, \varphi)(\psi(\alpha\alpha^*), \varphi)$ (by 2.5) and that ψ is symmetric.

On the other hand, if ψ is extendable, then let $\tilde{\psi}(\alpha + ce) = \tilde{\psi}_{x_0}(\alpha + ce) = \psi(\alpha) + cx_0$. Since $(\tilde{\psi}([\alpha + ce][\alpha + ce]^*), \varphi) = (\psi(\alpha\alpha^*), \varphi) + 2\operatorname{Re} \bar{c}(\psi(\alpha), \varphi) + |c|^2(x_0, \varphi)$, we have $(\tilde{\psi}([\alpha + ce][\alpha + ce]^*), \varphi) \geq (\psi(\alpha\alpha^*), \varphi) - 2|c| |(\psi(\alpha), \varphi)| + |c|^2(x_0, \varphi) \geq [(\psi(\alpha\alpha^*), \varphi)^{1/2} - |c|(x_0, \varphi)^{1/2}]^2 \geq 0$ (as ψ is extendable). Thus, $\tilde{\psi}$ is Φ -positive.

PROPOSITION 2.10. If there is an approximate identity $\{e_n\}$ in A , then a continuous Φ -positive mapping ψ is almost extendable.

Proof: Since $(\psi(\alpha^*), \phi) = \lim_{n \rightarrow \infty} (\psi(e_n \alpha^*), \phi) = \lim_{n \rightarrow \infty} \overline{(\psi(\alpha e_n^*), \phi)} = \overline{(\psi(\alpha), \phi)}$, ψ is symmetric and, since $|B_\phi(e_n, \alpha)|^2 \leq B_\phi(e_n, e_n) B_\phi(\alpha, \alpha) \leq \|\psi\| \|\phi\| B_\phi(\alpha, \alpha) = \|\psi\| \|\phi\| (\psi(\alpha\alpha^*), \phi)$, ψ is almost extendable.

In order to prove the extension of Bochner's theorem, we require a condition on the family Φ defining the cone of "positive" elements. As we shall see, the essential point is to deduce an estimate of the form $\|\psi(\alpha)\|^2 \leq k \|\psi(\alpha\alpha^*)\|$ from estimates of the form $|(\psi(\alpha), \phi)|^2 \leq d \|\phi\|^2 \|\psi\| \|\psi(\alpha\alpha^*)\|$ (ψ almost extendable) or $|(\psi(\alpha), \phi)|^2 \leq \|\phi\|^2 x_\phi \|\psi(\alpha\alpha^*)\|$ (ψ extendable). The following definition allows us to do this.

DEFINITION 2.11. The family Φ is full if there is a $\rho > 0$ such that

$$(2.12) \quad \begin{aligned} \|x\| &\leq \rho \sup_{\substack{\phi \in \Phi \\ \phi \neq 0}} \{ |(\psi(x), \phi)| / \|\phi\| \} \end{aligned}$$

for all x in X^+

We now have

LEMMA 2.13. If A has a unit e , if the involution on A is continuous, and if Φ is full, then every Φ -positive mapping ψ is continuous and almost extendable.

Proof: Suppose first that α is a Hermitian element of A with $\|\alpha\| \leq 1$. The binomial series $(1-t)^{1/2} = 1 - \frac{t}{2} - \frac{t^2}{2^2 2!} - \dots$ converges absolutely for

⁺This could be replaced by the following: Φ is full relative to ψ if there is a $\rho > 0$ such that $\|\psi(\alpha)\| \leq \rho \sup_{\substack{\phi \in \Phi \\ \phi \neq 0}} \{ |(\psi(\alpha), \phi)| / \|\phi\| \}$ for all α in A .

$|t| \leq 1$ and so the series $e - \frac{\alpha}{2} - \frac{\alpha^2}{2^2 2!} - \dots$ converges absolutely in A .

Since the involution is continuous, the sum β of this series is a Hermitian element of A with $\beta\beta^* = \beta^2 = e - \alpha$. It follows that $(\psi(e - \alpha), \phi) = (\psi(\beta\beta^*), \phi) \geq 0$ and hence, that $(\psi(e), \phi) \geq (\psi(\alpha), \phi)$. Replacing α by $-\alpha$, we have $(\psi(e), \phi) \geq (\psi(-\alpha), \phi)$. But $(\psi(\alpha), \phi)$ is real (since α is Hermitian) and so $|(\psi(\alpha), \phi)| \leq \|\phi\| \|\psi(e)\|$. Since ϕ is full, $\|\psi(\alpha)\| \leq \rho \|\psi(e)\|$.

Now, if α is any element of A , then $\alpha = \frac{1}{2}(\alpha + \alpha^*) - \frac{i}{2}(i(\alpha - \alpha^*))$. Since the involution is continuous, there is a $c > 0$ such that $\|\alpha^*\| \leq c\|\alpha\|$ and so, if $\|\alpha\| \leq 2/c+1$, then $\|(\alpha + \alpha^*)/2\| \leq 1$ and $\|i(\alpha - \alpha^*)/2\| \leq 1$. It follows that $\|\psi(\alpha)\|^2 \leq 2\rho \|\psi(e)\|$ for all α in A with $\|\alpha\| \leq 2/c+1$. Thus, ψ is bounded and therefore continuous.

Since $|(\psi(\alpha), \phi)|^2 \leq (\psi(e), \phi)(\psi(\alpha\alpha^*), \phi) \leq \|\psi\| \|\phi\| (\psi(\alpha\alpha^*), \phi)$, ψ is almost extendable.

COROLLARY 2.14. If the involution on A is continuous, if ϕ is full, and if ψ is ϕ -positive and extendable, then ψ is continuous and almost extendable.

Proof: Apply proposition 2.9 and the lemma.

Let G be a σ -finite locally compact abelian group and let $A = L_1(G, \mathbb{C})$. The involution on $L_1(G, \mathbb{C})$ is given by $\alpha^*(g) = \overline{\alpha(-g)}$ and is continuous since $L_1(G, \mathbb{C})$ is semi-simple. Observe that if ϕ is full and ψ is a ϕ -positive mapping of $L_1(G, \mathbb{C})$ into X , then ψ is continuous and almost extendable if ψ is extendable (corollary 2.14) and conversely, ψ is almost extendable if ψ is continuous (proposition 2.10).

Now let us introduce the following

DEFINITION 2.15 Let p be an element of $L_\infty(G, X)$. The mapping p is Φ -positive definite if

$$(2.16) \quad \sum_{n=1}^N \sum_{m=1}^N c_n \overline{c_m} (p(g_n - g_m), \varphi) \geq 0$$

for any integer N , any c_1, \dots, c_N in C , any g_1, \dots, g_N in G , and all φ in Φ . The mapping p is integrally Φ -positive definite if

$$(2.17) \quad \left(\int_G \int_G \alpha(g) \overline{\alpha(g')} p(g - g') d\mu d\mu, \varphi \right) \geq 0$$

for all α in $L_1(G, C)$ and all φ in Φ .

We then have

PROPOSITION 2.18. Let p be a continuous element of $L_\infty(G, X)$. Then p is Φ -positive definite if and only if p is integrally Φ -positive definite.

Proof: If p is Φ -positive definite, then p is integrally Φ -positive definite by a result of Naimark ([6], p. 397). Conversely, if p is integrally Φ -positive definite, then there is a continuous positive definite function f_φ mapping G into C such that $f_\varphi(g) = (p(g), \varphi)$ locally almost everywhere on G ([6], p. 397) for each φ in Φ . Since $(p(\cdot), \varphi)$ is continuous, $f_\varphi(\cdot) = (p(\cdot), \varphi)$ everywhere and hence, p is Φ -positive definite.

Now it is a fact that ψ is a bounded weakly compact linear map of $L_1(G, C)$ into X with separable range if and only if there is a p with (essentially) weakly compact range in $L_\infty(G, X)$ such that

$$(2.19) \quad \psi(\alpha) = \int_G \alpha(g) p(g) d\mu$$

for all α in $L_1(G, C)$ ([9], p. 279, or [7], p. 507). Moreover, $\|\psi\| = \|p\|_\infty$. The fact that the weakly compact maps in $\mathcal{L}(L_1(G, C), X)$ are essentially the same as the functions with (essentially) weakly compact range in $L_\infty(G, X)$ will allow us to relate the notion of Φ -positivity to the notions of Φ -positive definiteness and integral Φ -positive definiteness.

LEMMA 2.20. Let Φ be full. If ψ is a weakly compact linear mapping of $L_1(G, C)$ into X which is Φ -positive and extendable, then there is an (essentially unique) integrally Φ -positive p in $L_\infty(G, X)$ such that

$$(2.21) \quad \psi(\alpha) = \int_G \alpha(g) p(g) d\mu$$

for all α in $L_1(G, C)$. Conversely, if p is an integrally Φ -positive definite element of $L_\infty(G, X)$ and ψ is given by 2.21, then ψ is Φ -positive and almost extendable.

Proof: Assume that ψ is given. In view of [9], p. 279, the mapping p exists and we need only show that p is integrally Φ -positive definite.

But

$$(2.22) \quad \psi(\alpha\alpha^*) = \int_G \int_G \alpha(g-g') \overline{\alpha(-g')} p(g) d\mu d\mu = \int_G \int_G \alpha(g) \overline{\alpha(g')} p(g-g') d\mu d\mu$$

by virtue of the Fubini and Tonelli theorems and the invariance of Haar measure. Conversely, given p , we simply note that $\psi(\alpha\alpha^*)$ is determined

by 2.22 in order to prove that ψ is Φ -positive. Moreover, since ψ is continuous, ψ is almost extendable by proposition 2.10.

3. Bochner's Theorem for Algebras

Before proving the generalization of Bochner's theorem to maps of A into X , we recall the following

DEFINITION 3.1. Let S be a locally compact topological space and let $\Sigma(S)$ be the Borel field of S . A vector measure ν is a weakly countably additive set function taking values in X . The vector measure ν is weakly regular if the scalar measures $(\nu(\cdot), \phi)$ are regular⁺ for all ϕ in X^* . The vector measure ν is Φ -positive if $(\nu(E), \phi) \geq 0$ for all ϕ in Φ and E in $\Sigma(S)$. A set function ν^{**} mapping $\Sigma(S)$ into X^{**} is weak-* -regular if $(\phi, \nu^{**}(\cdot))$ is a regular scalar measure for all ϕ in X^* . The set function ν^{**} is Φ -positive if $(\phi, \nu^{**}(E)) \geq 0$ for all ϕ in Φ and E in $\Sigma(S)$.

We now have

THEOREM 3.2. Let A be a self-adjoint commutative Banach algebra whose involution satisfies the condition $(\hat{\alpha}^*) = \bar{\alpha}$ (e.g. A semi-simple) and let Φ be a full family. If ψ is a mapping of A into X , then ψ is Φ -positive and almost extendable if and only if there is a set function ν^{**} mapping $\Sigma(\mathcal{M})$ into X^{**} such that (i) ν^{**} is weak-* -regular, (ii) ν^{**} is Φ -positive, (iii) ν^{**} is finite i.e. $\|\nu^{**}\|(\mathcal{M}) < \infty$, (iv) the mapping $T_{\nu^{**}}$

⁺ A scalar measure μ is regular if given $\epsilon > 0$ and $E \in \Sigma(S)$ with $\|\mu\|(E) < \infty$, then there is a compact $K \subseteq E$ and an open $O \supseteq E$ such that $\|\mu\|(O-K) < \epsilon$.

of X^* into the scalar measures on \mathcal{M} given by $T_{v^{**}}(\varphi) = (v^{**}(\cdot), \varphi)$ is continuous in the X and $C_0(\mathcal{M})^+$ topologies in these spaces respectively, and (v)

$$(3.3) \quad (\psi(\alpha), \varphi) = \int_{\mathcal{M}} \hat{\alpha}(M) d(v^{**}, \varphi)$$

for all α in A and all φ in X^* .

Proof: Suppose first that ψ is Φ -positive and almost extendable. Then ψ is continuous. Let $\hat{\psi}$ be the map of \hat{A} into X given by $\hat{\psi}(\hat{\alpha}) = \psi(\alpha)$. Then $\|\hat{\psi}(\hat{\alpha})\| = \|\psi(\alpha)\|$ and $|(\psi(\alpha), \varphi)|^2 \leq d\|\psi\|\|\varphi\|(\psi(\alpha\alpha^*), \varphi) \leq d\|\psi\|\|\varphi\|^2\|\psi(\alpha\alpha^*)\|$ for all φ in Φ (since ψ is almost extendable). Since Φ is full, there is a $\rho > 0$ such that $\|\psi(\alpha)\| \leq \rho \sup_{\substack{\varphi \in \Phi \\ \varphi \neq 0}} \{ |(\psi(\alpha), \varphi)| / \|\varphi\| \}$. Thus, there is a

constant $k (= \rho^2 d\|\psi\|)$ such that

$$(3.4) \quad \|\psi(\alpha)\|^2 \leq k\|\psi(\alpha\alpha^*)\|$$

for all α in A . It follows that $\|\psi(\alpha)\|^2 \leq k\|\psi(\alpha\alpha^*)\| \leq k^{1+1/2}\|\psi([\alpha\alpha^*]^2)\|^{1/2} \dots \leq k^2\|\psi\|_0^2\|\hat{\alpha}\|_\infty^2$ and hence, that $\hat{\psi}$ is a bounded linear map.

Since A is self-adjoint and commutative, \hat{A} is dense in $C_0(\mathcal{M})$ and $\hat{\psi}$ can, therefore, be extended to $C_0(\mathcal{M})$. Let $\hat{\psi}_e$ denote the extension of $\hat{\psi}$ to $C_0(\mathcal{M})$. We claim that there is a weak-* regular set function v^{**} on $\Sigma(\mathcal{M})$ such that

⁺If \mathcal{M} is compact, then $C_0(\mathcal{M})$ is the set of all continuous complex valued functions on \mathcal{M} . If \mathcal{M} is locally compact but not compact, then $C_0(\mathcal{M})$ is the set of all continuous complex valued functions on \mathcal{M} which "vanish at infinity".

$$(3.5) \quad (\hat{\psi}_e(f), \varphi) = \int_{\mathcal{M}} f(M) d(\nu^{**}, \varphi)$$

for all f in $C_0(\mathcal{M})$ and φ in X^* .

To verify this claim, we let $M(\mathcal{M})$ be the space of all complex valued regular measures μ on \mathcal{M} for which $\|\mu\|$ is finite ([2]). Note that $C_0(\mathcal{M})^* = M(\mathcal{M})$ by the Riesz representation theorem. If $E \in \Sigma(\mathcal{M})$, then let T_E be the element of $C_0(\mathcal{M})^{**}$ defined by

$$(3.6) \quad T_E(\mu) = \mu(E), \quad \mu \in M(\mathcal{M})$$

Now define a set function ν^{**} of $\Sigma(\mathcal{M})$ into X^{**} by setting

$$(3.7) \quad \nu^{**}(E) = \hat{\psi}_e^{**}(T_E)$$

for E in $\Sigma(\mathcal{M})$. We show that ν^{**} is weak- $*$ -regular. If φ is an element of X^* , then $\hat{\psi}_e^*(\varphi)$ is, by the Riesz representation theorem, a measure μ_φ in $M(\mathcal{M})$. But

$$(3.8) \quad \mu_\varphi(E) = T_E(\mu_\varphi) = T_E(\hat{\psi}_e^*(\varphi)) = \hat{\psi}_e^{**}(T_E)(\varphi) = (\nu^{**}(E), \varphi)$$

and so, the set function ν^{**} is weak- $*$ -regular. Moreover, since $\hat{\psi}_e^*(\varphi) = (\nu^{**}(\cdot), \varphi)$ by 3.8, the mapping $T_{\nu^{**}}$ satisfies (iv). Also, $(\hat{\psi}_e(f), \varphi) = \hat{\psi}_e^*(\varphi)(f) = \int_{\mathcal{M}} f(M) d\mu_\varphi = \int_{\mathcal{M}} f(M) d(\nu^{**}, \varphi)$ for f in $C_0(\mathcal{M})$ so that 3.3 is satisfied. It is easy to check that $\|\nu^{**}\|(\mathcal{M}) = \|\hat{\psi}_e\|$ ([7], p. 492) and so, (iii) is satisfied.

All that remains to establish the first half of the theorem is to prove that ν^{**} is Φ -positive. If f is an element of $C_0(\mathcal{M})$ with $f(M) \geq 0$ for all M , then $f^{1/2}$ is in $C_0(\mathcal{M})$ and there is a sequence $\{\alpha_n\}$ in A such that $\lim_{n \rightarrow \infty} \hat{\alpha}_n = f^{1/2}$. Since $(\hat{\psi}(\alpha_n \alpha_n^*), \varphi) = \int_{\mathcal{M}} |\hat{\alpha}_n(M)|^2 d(\nu^{**}, \varphi)$, it follows that if φ is an element of Φ , then $0 \leq (\psi(\alpha_n \alpha_n^*), \varphi) = \int_{\mathcal{M}} |\hat{\alpha}_n(M)|^2 d(\nu^{**}, \varphi)$ and hence, by taking limits, that

$$(7.9) \quad \int_{\mathcal{M}} f(M) d(\nu^{**}, \varphi) \geq 0$$

for all φ in Φ and all f in $C_0(\mathcal{M})$ with $f(\cdot) \geq 0$. But $(\nu^{**}(\cdot), \varphi)$ when restricted to the Baire sets in \mathcal{M} is a Baire measure, and as such, is positive. The Baire measure can be extended to a unique regular Borel measure ([9]) which must (by uniqueness) be $(\nu^{**}(\cdot), \varphi)$. It follows that ν^{**} is Φ -positive.

Now suppose that ν^{**} is given. Since the mapping $T_{\nu^{**}}$ is continuous in the X and $C_0(\mathcal{M})$ topologies, the linear mapping $\varphi \rightarrow \int_{\mathcal{M}} f(M) d(\nu^{**}, \varphi)$ is, for each fixed f in $C_0(\mathcal{M})$, continuous in the X -topology of X^* and is, therefore, generated by an element x_f of X . Thus, the mapping $\hat{\psi}_e$ of $C_0(\mathcal{M})$ into X given by $\hat{\psi}_e(f) = x_f$ is a bounded linear map of $C_0(\mathcal{M})$ into X . If α is an element of A , then let $\psi(\alpha) = \hat{\psi}_e(\hat{\alpha})$. Since $\|\psi(\alpha)\| = \|\hat{\psi}_e(\hat{\alpha})\| \leq \|\hat{\psi}_e\| \|\hat{\alpha}\|_{\infty} \leq \|\hat{\psi}_e\| \|\alpha\|$, ψ is a continuous linear map. If φ is an element of Φ , then $(\psi(\alpha \alpha^*), \varphi) = \int_{\mathcal{M}} |\hat{\alpha}(M)|^2 d(\nu^{**}, \varphi) \geq 0$ and $(\psi(\alpha^*), \varphi) = \int_{\mathcal{M}} \overline{\hat{\alpha}(M)} d(\nu^{**}, \varphi) = \overline{\int_{\mathcal{M}} \hat{\alpha}(M) d(\nu^{**}, \varphi)} = \overline{(\psi(\alpha), \varphi)}$ so that ψ is Φ -positive. Also, $|(\psi(\alpha), \varphi)|^2 \leq [\int_{\mathcal{M}} |\hat{\alpha}(M)|^2 d(\nu^{**}, \varphi)]$

$[\int_{\mathcal{M}} 1^2 d(\nu^{**}, \varphi)] \leq (\psi(\alpha\alpha^*), \varphi)(\nu^{**}(\mathcal{M}), \varphi) \leq \|\nu^{**}\|(\mathcal{M})\|\varphi\|(\psi(\alpha\alpha^*), \varphi) \leq$
 $\max\{1, \|\nu^{**}\|(\mathcal{M})/\|\psi\|\}\|\psi\|\|\varphi\|(\psi(\alpha\alpha^*), \varphi)$ so that ψ is almost extendable.

COROLLARY 3.10. Let A be a self-adjoint commutative Banach algebra with $(\hat{\alpha}^*) = \bar{\hat{\alpha}}$ and let Φ be a full family. If ψ is Φ -positive and almost extendable and if $\hat{\psi}$ is weakly compact, then there is a weakly regular Φ -positive vector measure ν on $\Sigma(\mathcal{M})$ such that

$$(3.11) \quad \psi(\alpha) = \int_{\mathcal{M}} \hat{\alpha}(M) d\nu$$

for all α in A . Conversely, if ν is a weakly regular Φ -positive vector measure and ψ is given by (3.11), then ψ is Φ -positive and almost extendable and $\hat{\psi}$ is weakly compact.

Proof: Suppose that ψ is given. Since $\hat{\psi}$ is weakly compact, $\hat{\psi}_e^+$ is weakly compact and so, $\hat{\psi}_e^{**}(C_0(\mathcal{M})^{**})$ is contained in the natural imbedding of X in X^{**} . Thus, the set function ν^{**} given by 3.7 may be identified with a mapping ν of $\Sigma(\mathcal{M})$ into X . In that case, $(\nu(\cdot), \varphi)$ is an element of $M(\mathcal{M})$ for all φ in X^* . It follows that $\nu(\cdot)$ is a weakly regular vector measure (as $\nu(\cdot)$ is weakly countably additive). Clearly ν is Φ -positive. Moreover, since $(\psi(\alpha), \varphi) = \int_{\mathcal{M}} \hat{\alpha}(M) d(\nu, \varphi) = (\int_{\mathcal{M}} \hat{\alpha}(M) d\nu, \varphi)$ for all φ in X^* , 3.11 is satisfied.

On the other hand, if ν is given and ψ is defined by 3.11 (note that $\hat{\alpha}(\cdot)$ is bounded and continuous), then ψ is Φ -positive and

⁺We use the notation of the proof of the theorem.

almost extendable. In fact, $\|\psi(\alpha)\| \leq \|\hat{\alpha}\|_{\infty} \|v\|(\mathcal{M}) \leq \|\alpha\| \|v\|(\mathcal{M})$ and $|(\psi(\alpha), \varphi)|^2 \leq (\psi(\alpha\alpha^*), \varphi)(v(\mathcal{M}), \varphi)$ so that ψ is extendable ($x_0 = v(\mathcal{M}) \in X$). Thus, to complete the proof we need only show that $\hat{\psi}$ is weakly compact.

Now, $\hat{\psi}$ is clearly linear and, since $\|\hat{\psi}(\hat{\alpha})\| = \|\psi(\alpha)\| \leq \|v\|(\mathcal{M}) \|\hat{\alpha}\|_{\infty}$, $\hat{\psi}$ is continuous. Let $\hat{\psi}_e$ be the mapping of $C_0(\mathcal{M})$ into X defined by $\hat{\psi}_e(f) = \int_{\mathcal{M}} f(M) dv$. Thus, it will be enough to prove that $\hat{\psi}_e$ is weakly compact.

If φ is an element of X^* , then $\hat{\psi}_e^*(\varphi) = (v(\cdot), \varphi)$ is an element of $M(\mathcal{M})$. Since the set $\{(v(\cdot), \varphi) : \varphi \in X^*, \|\varphi\| \leq 1\}$ is weakly sequentially compact as a subset of the space of scalar measures and since v is weakly regular, $\hat{\psi}_e^*$ is a weakly compact mapping. It follows that $\hat{\psi}_e$ is weakly compact and the corollary is established.

COROLLARY 3.12. If v satisfies the conditions of corollary 3.10 and ψ is given by 3.11, then ψ is extendable. Conversely, if ψ is extendable (rather than almost extendable) and if the involution on A is continuous (e.g. A semi-simple), then a v satisfying the conditions of corollary 3.10 exists (the other hypotheses of corollary 3.10 are, of course, assumed).

Proof: The first assertion was established in the course of the proof of corollary 3.10. The second assertion is an immediate consequence of corollary 2.14.

COROLLARY 3.13. If X is weakly complete, if A and Φ satisfy the conditions of corollary 3.10, and if ψ is Φ -positive and almost extendable, then $\hat{\psi}$ is weakly compact.

Proof: By the argument given in the proof of theorem 3.2, $\hat{\psi}$ is a continuous linear map. If A has a unite, then \mathcal{M} is compact. Since \hat{A} is dense in $C_0(\mathcal{M})$, we may extend $\hat{\psi}$ to a continuous linear map $\hat{\psi}_e$ of $C_0(\mathcal{M})$ into X . As X is weakly complete, $\hat{\psi}_e$ is weakly compact ([7], p. 494) and a fortiori so is $\hat{\psi}$. If A does not have a unit, then we extend A to $\tilde{A} = A \oplus C$. Letting x_0 be an element of X , we extend ψ to a mapping $\tilde{\psi}$ of \tilde{A} into X by setting $\tilde{\psi}(\alpha + \lambda e) = \psi(\alpha) + \lambda x_0$. Then $\hat{\tilde{\psi}}(\hat{\alpha} + \lambda \hat{e}) = \hat{\psi}(\hat{\alpha}) + \lambda x_0$ is a bounded linear map of $\hat{\tilde{A}}$ into X . It follows that $\hat{\tilde{\psi}}$ is weakly compact and hence, that $\hat{\psi}$ is weakly compact.

4. Bochner's Theorem on a Group

Let G be a σ -finite locally compact abelian group and let $A = L_1(G, C)$. The involution on A is given by $\alpha^*(g) = \overline{\alpha(-g)}$ and is continuous. Let X be a Banach space and let Φ be a full family. We shall prove a generalization of Bochner's theorem for integrally Φ -positive definite mappings p in $L_\infty(G, X)$ by combining lemma 2.20 with theorem 3.2 and its corollaries. We have

THEOREM 4.1. (A) If ν is a weakly regular Φ -positive vector measure defined on $\Sigma(\hat{G})$ (the Borel field of the dual group \hat{G}) and if

$$(4.2) \quad p(g) = \int_{\hat{G}} \overline{\gamma(g)} d\nu$$

then p is an integrally Φ -positive definite element of $L_\infty(G, X)$.

(B) If p is an integrally Φ -positive definite element of $L_\infty(G, X)$, then there is a set function ν^{**} mapping $\Sigma(\hat{G})$ into X^{**} such

that (i) ν^{**} is weak*-regular, Φ -positive, and finite, (ii) the map $T_{\nu^{**}}$ given by $T_{\nu^{**}}(\varphi) = (\nu^{**}(\cdot), \varphi)$ is continuous in the X topology of X^* and the $C_0(\hat{G})$ topology of $M(\hat{G})$, and (iii)

$$(4.3) \quad (p(g), \varphi) = \int_{\hat{G}} \overline{(\gamma, g)} d(\nu^{**}, \varphi)$$

for all φ in X^* and (almost) all g in G .

Proof: (A) Let $p(\cdot)$ be given by 4.2. Suppose, for the moment, that $p(\cdot)$ is measurable. Then p is in $L_{\infty}(G, X)$ since $\|p(g)\| \leq \|\nu\|(\hat{G})$ for all g . Let $\psi(\alpha) = \int_G \alpha(g) p(g) d\mu$ for α in $L_1(G, \mathbb{C})$. Then

$$\begin{aligned} (\psi(\alpha), \varphi) &= \int_G \alpha(g) \int_{\hat{G}} \overline{(\gamma, g)} d(\nu, \varphi) d\mu \\ &= \int_{\hat{G}} \int_G \alpha(g) \overline{(\gamma, g)} d\mu d(\nu, \varphi) \\ &= \int_{\hat{G}} \hat{\alpha}(\gamma) d(\nu, \varphi) = (\int_{\hat{G}} \hat{\alpha}(\gamma) d\nu, \varphi) \end{aligned}$$

for all φ in X^* by the Fubini and Tonelli theorems. Since \hat{G} and \mathcal{M} can be identified ([2] or [3]), we have $\psi(\alpha) = \int_{\mathcal{M}} \hat{\alpha}(M) d\nu$ (as ν may be viewed as a measure on \mathcal{M}). But then (corollary 3.10) ψ is Φ -positive and extendable (corollary 3.12). The result follows immediately from 2.22 of lemma 2.20.

Thus, to complete the proof of (A), we need only show that p is measurable. To do this it will be sufficient to show that, for any set $F \subset G$ with $\mu(F) < \infty$, $P_F(\cdot) = \chi_F(\cdot) p(\cdot)$ is the limit in measure of a sequence of simple functions where χ_F is the characteristic function of F .

Since ν is weakly regular, there is a finite, positive, regular scalar measure λ such that $\|\nu\|(E) \rightarrow 0$ if and only if $\lambda(E) \rightarrow 0$ where $\|\nu\|(E)$ is the semi-variation of ν on E ([7]). Therefore, given $\eta > 0$, there is a $\xi > 0$ such that if $\lambda(\hat{G}-K)^+ < \xi$, then $\|\nu\|(\hat{G}-K) < \eta/4$ for K compact in \hat{G} . Since λ is finite and regular, there is a compact set $K \subset \hat{G}$ for which $\lambda(\hat{G}-K) < \xi$ and hence for which $\|\nu\|(\hat{G}-K) < \eta/4$. Let $\eta_1 = \eta/2\|\nu\|(\hat{G})$ and let $N(g; K, \eta_1) = \{g' \in G: |1-(r, g')| < \eta_1, r \in K\} + g$. Then $N(g; K, \eta_1)$ is an open neighborhood of g in G .

Now G is σ -finite and so there is an increasing sequence of sets G_n with $\mu(G_n) < \infty$ and $\bigcup G_n = G$. Moreover, since Haar measure is regular, given $\epsilon > 0$ there is a compact set $L_n \subseteq G_n$ such that $\mu(G_n - L_n) < \epsilon$. The sets $N(g; K, \eta_1)$, $g \in L_n$, form an open cover of L_n . Thus there are g_1, \dots, g_{M_n} in L_n such that $L_n \subseteq \bigcup_{i=1}^{M_n} N(g_i; K, \eta_1)$. Let $N_1^n = N(g_1; K, \eta_1)$ and $N_{i+1}^n = N(g_{i+1}; K, \eta_1) - (N(g_1; K, \eta_1) \cup \dots \cup N(g_i; K, \eta_1))$. Then $L_n \subseteq \bigcup_{i=1}^{M_n} N_i^n$ and the union is disjoint. Let p_0 be defined on L_n by $p_0(g) = p(g_i)$ if $g \in N_i^n$ and let $p_n^{\epsilon, \eta}(\cdot)$ be given by

$$(4.4) \quad p_n^{\epsilon, \eta}(g) = \begin{cases} p_0(g) & g \in L_n \\ 0 & g \notin L_n \end{cases}$$

Then $p_n^{\epsilon, \eta}(\cdot)$ is a simple function and we claim that

$$(4.5) \quad \mu^*({g \in G_n: \|p(g) - p_n^{\epsilon, \eta}(g)\| > \eta}) < \epsilon$$

where $\mu^*(E) = \inf_{E_1 \supseteq E} \mu(E_1)$. For if g is in L_n , then $\|p(g) - p_n^{\epsilon, \eta}(g)\| =$

*Here $\hat{G}-K$ is the complement of K .

$\|p(g) - p_0(g)\| = \|\int_{\hat{G}} \overline{(\gamma, g)} [1 - \overline{(\gamma, g_1 - g)}] d\nu\| \leq \|\int_{\hat{G}-K} \overline{(\gamma, g)} [1 - \overline{(\gamma, g_1 - g)}] d\nu\| +$
 $\|\int_K \overline{(\gamma, g)} [1 - \overline{(\gamma, g_1 - g)}] d\nu\| \leq \frac{\eta}{2} + \eta_1 \|v\|(\hat{G}) = \eta$ (for some i) so that $\{g \in G_n:$
 $\|p(g) - p_n^{\epsilon, \eta}(g)\| > \eta\} \subseteq G_n - L_n$. It follows that 4.5 holds. Let $p_n(g) =$
 $p_n^{1/n, 1/n}(g)$ so that p_n is a simple function.

Now suppose that a is any positive number. We show that

$$(4.6) \quad \lim_{n \rightarrow \infty} \mu^* (\{g \in F: \|p(g) - p_n(g)\| > a\}) = 0$$

for any $F \subset G$ with $\mu(F) < \infty$. So let $\epsilon > 0$ be given. Then there is an $n_0 \geq \max(1/a, 2/\epsilon)$ such that $\mu(F \cap (G - G_n)) < \epsilon/2$ for $n \geq n_0$. It follows that

$$\begin{aligned}
 \mu^* (\{g \in F: \|p(g) - p_n(g)\| > a\}) &\leq \mu^* (\{g \in F \cap G_n: \|p(g) - p_n(g)\| > a\}) + \epsilon/2 \\
 &\leq \mu^* (\{g \in F \cap G_n: \|p(g) - p_n(g)\| > 1/n\}) + \epsilon/2 \\
 &\leq 1/n + \epsilon/2 \leq \epsilon
 \end{aligned}$$

for $n \geq n_0$. In other words, p_n converges to p in measure on F . The proof of (A) is now complete.

(B) Let $\psi(\alpha) = \int_G \alpha(g) p(g) d\mu$. Then ψ is Φ -positive and almost extendable by lemma 2.20. It follows from theorem 3.2 that there is a set function ν^{**} on $\Sigma(\mathcal{M})$ such that (i) and (ii) are satisfied and

$$(4.7) \quad (\psi(\alpha), \varphi) = \int_{\mathcal{M}} \hat{\alpha}(M) d(\nu^{**}, \varphi)$$

for all φ in X^* . Since \hat{G} and \mathcal{M} can be identified, ν^{**} may be viewed as a set function on $\Sigma(\hat{G})$ and

$$(4.8) \quad (\psi(\alpha), \varphi) = \int_{\hat{G}} [\int_G \alpha(g) \overline{(\gamma, g)} d\mu] d(\nu^{**}, \varphi)$$

for all φ in X^* . Application of the Fubini and Tonelli theorems then yields

$$(4.9) \quad \int_G \alpha(g) (p(g), \varphi) d\mu = (\psi(\alpha), \varphi) = \int_G \alpha(g) q_{\varphi}(g) d\mu$$

where $q_{\varphi}(g) = \int_{\hat{G}} \overline{(\gamma, g)} d(\nu^{**}, \varphi)$. Since $(p(\cdot), \varphi)$ and $q_{\varphi}(\cdot)$ are in $L_{\infty}(G, \mathbb{C})$, we have $\|(p(\cdot), \varphi) - q_{\varphi}(\cdot)\|_{\infty} = 0$ for all φ in X^* . In other words, 4.3 holds. The proof of (B) is now complete.

REMARK 4.10. Since $\overline{(\gamma, g)} = (-\gamma, g)$ and since the measure ν_1 (or the set function ν_1^{**}) given by $\nu_1(E) = \nu(-E)$ (or $\nu_1^{**}(E) = \nu^{**}(-E)$) has the same properties as ν (or ν^{**}), $p(g)$ is given by $p(g) = \int_{\hat{G}} (\gamma, g) d\nu_1$ (or satisfies $(p(g), \varphi) = \int_{\hat{G}} (\gamma, g) d(\nu_1^{**}, \varphi)$). [This agrees with convention in the scalar case.]

We observe that if the hypotheses of (A) are satisfied and $\psi(\alpha) = \int_G \alpha(g) p(g) d\mu$, then the mapping $\hat{\psi}$ of \hat{A} into X given by $\hat{\psi}(\hat{\alpha}) = \psi(\alpha)$ is weakly compact (corollary 3.10). Note also that if X is weakly complete and $p(\cdot)$ is an integrally Φ -positive definite element of $L_{\infty}(G, X)$, then $\hat{\psi}$ is weakly compact. This leads to

COROLLARY 4.11. If X is weakly complete and if p is an integrally Φ -

positive definite element of $L_\infty(G, X)$, then there is a weakly regular Φ -positive vector measure ν on $\Sigma(\hat{G})$ such that

$$(4.12) \quad (p(g), \varphi) = \left(\int_{\hat{G}} \hat{\alpha}(r, g) d\nu, \varphi \right)$$

for all φ in X^* and (almost) all g in G . If, in addition, Φ is countable, then

$$(4.13) \quad p(g) = \int_{\hat{G}} \hat{\alpha}(r, g) d\nu$$

for (almost) all g in G .

Proof. The first assertion follows from corollary 3.12. On the other hand, if $\Phi = \{\varphi_i\}$ is countable, then there is a μ -null set N such that

$$(p(g) - q(g), \varphi) = 0 \text{ for all } \varphi \text{ in } \Phi \text{ and } g \notin N \text{ where } q(g) = \int_{\hat{G}} \hat{\alpha}(r, g) d\nu.$$

But then $\|p(g) - q(g)\| \leq \rho \sup_{\substack{\varphi \in \Phi \\ \varphi \neq 0}} \{|(p(g) - q(g), \varphi)| / \|\varphi\|\} = 0$ for $g \notin N$. It

follows immediately that $\|p(\cdot) - q(\cdot)\|_\infty = 0$, i.e. that 4.13 holds.

In order to state our final corollary we require

DEFINITION 4.14. The element p of $L_\infty(G, X)$ is dominated if there exists a finite regular positive Borel measure λ such that

$$(4.15) \quad \left\| \int_{\hat{G}} \alpha(g) p(g) d\mu \right\| \leq \int_{\hat{G}} |\hat{\alpha}(r)| d\lambda$$

for all α in $L_1(G, C)$, where $\hat{\alpha}$ is the Fourier transform of α .

COROLLARY 4.16. Assume Φ is countable. Then p is a dominated integrally Φ -positive definite element of $L_\infty(G, X)$ if and only if there exists a weakly regular Φ -positive vector measure ν of finite variation mapping $\Sigma(\hat{G})$ into X such that

$$(4.17) \quad p(g) = \int_{\hat{G}} (r, g) d\nu.$$

Proof. We have only to note that there exists an isomorphism between the set of weakly regular vector measures $\nu: \Sigma(\hat{G}) \rightarrow X$ with finite variation and the set of bounded linear operators $T: C_0(\hat{G}) \rightarrow X$ for which there exists a finite regular positive Borel measure λ such that $\|T(f)\| \leq \int_{\hat{G}} |f(r)| d\lambda$. This isomorphism is given by $T(f) = \int_{\hat{G}} f(r) d\nu$, ([9], p. 380, or [11]). Now using theorem 4.1 (B) we have, if we assume the existence of p , that $(\int_{\hat{G}} f(r) d\nu, \phi) = \int_{\hat{G}} f(r) d(\nu^{**}, \phi)$ for any f in $C_0(\hat{G})$, ϕ in X^* . But $C_0(\hat{G})^* = M(\hat{G})$, the space of regular complex valued measures defined on $\Sigma(\hat{G})$ of finite variation, and $(\phi, \nu^{**}), (\nu, \phi)$ are in $M(\hat{G})$. Thus, for any E in $\Sigma(\hat{G})$, $(\nu(E), \phi) = (\phi, \nu^{**}(E))$. Consider $\nu(E)$ as an element of X^{**} , then $\nu(E) = \nu^{**}(E)$ and so ν^{**} is actually a measure. From the countability of Φ we derive (4.17).

The converse follows immediately from theorem 4.1 (A).

5. Some Examples

We now give several examples of spaces to which the theory applies.

EXAMPLE 5.1. Let $X = L_1([0,1], \mathbb{C})$. Note that X is weakly complete. If $\Sigma = \Sigma([0,1])$ is the Borel field on $[0,1]$, then Σ is a separable metric space with respect to the usual metric $d(E, E') = \mu(E \Delta E')$ where $E \Delta E' =$

$(E-E') \cup (E'-E)$ is the symmetric difference of E and E' . Let $\{E_i\}$ be a countable dense set in Σ with $E_1 = [0,1]$. Let χ_i be the characteristic function of E_i and let ϕ_i be the element of X^* given by

$$(5.2) \quad (x(\cdot), \phi_i) = \int_0^1 \chi_i(s) x(s) ds$$

If $\Phi = \{\phi_i\}$, then K_Φ is the cone of (essentially) nonnegative functions. Note also that $\|\phi_i\| \leq 1$.

Now we claim that Φ is full. Set $x^+(s) = \max\{0, x(s)\}$ and $x^-(s) = \max\{0, -x(s)\}$ for real x in X . Then $x(s) = x^+(s) - x^-(s)$ and $|x(s)| = x^+(s) + x^-(s)$. Moreover, x^+ and x^- are nonnegative. Letting $\int_0^1 x_0(s) ds = \max\{\int_0^1 x^+(s) ds, \int_0^1 x^-(s) ds\}$ (i.e. $x_0 = x^+$ or x^- according to which integral is greater), we see that

$$(5.3) \quad \|x(\cdot)\|_1 \leq 2 \int_0^1 x_0(s) ds$$

for real x in X . Now, suppose, for example, that $x_0 = x^+$. Since x^+ is measurable, $(x^+)^{-1}([0, \infty)) = E$ is in Σ and $\int_0^1 x_0(s) ds = \int_E x^+(s) ds = \int_E x(s) ds$. As $\{E_i\}$ is dense in Σ , there is a sequence $\{E_{i,n}\}$ such that $d(E_{i,n}, E) \rightarrow 0$ as $n \rightarrow \infty$. But $|\int_{E_{i,n}} x(s) ds - \int_E x(s) ds| \leq \int_{E_{i,n} \Delta E} |x(s)| ds$ and

$\lim_{n \rightarrow \infty} \int_{E_{i,n} \Delta E} |x(s)| ds = 0$ as $\mu(E_{i,n} \Delta E) \rightarrow 0$ as $n \rightarrow \infty$ and x is in

$L_1([0,1], \mathbb{R})$. It follows that there is a sequence $\{\phi_{i,n}\}$ such that $\lim_{n \rightarrow \infty} (x, \phi_{i,n}) = \int_0^1 x_0(s) ds$ and hence, that $\int_0^1 x_0(s) ds \leq \sup_{\phi \in \Phi} |(x, \phi)|$. Now choose any x in

X . Then $x = x_1 + ix_2$ where $x_1(\cdot), x_2(\cdot)$ are real valued. But $\|\phi\| \leq 1$ for ϕ in Φ and so,

$$(5.4) \quad \|x\|_1 \leq \|x_1\|_1 + \|x_2\|_1 \leq 4 \sup_{\varphi \in \Phi} |(x, \varphi)| \leq 4 \sup_{\substack{\varphi \in \Phi \\ \varphi \neq 0}} \{ |(x, \varphi)| / \|\varphi\| \}$$

for all x in X .

EXAMPLE 5.5. Let $X = H$ be a separable Hilbert space. Fix an orthonormal basis $\{e_i\}$ in H . If $h \in H$, then $h = h_1 + ih_2$ where $h_1 = \sum \operatorname{Re}[\langle h, e_i \rangle] e_i$ and $h_2 = \sum \operatorname{Im}[\langle h, e_i \rangle] e_i$. An element h is real if $h = h_1$. Let H_0 be the set of all real elements h such that (i) $\|h\| \leq 1$, (ii) h is positive i.e. $\langle h, e_i \rangle \geq 0$ for all i , (iii) h is rational i.e. $\langle h, e_i \rangle$ is rational for all i , and (iv) h is finite i.e. only a finite number of components $\langle h, e_i \rangle$ of h are not zero. Since H^* can be identified with H , we let $\Phi = H_0$. In other words, if $\varphi \in \Phi$, then $(h, \varphi) = \langle h, k \rangle$ for some k in H_0 . The cone K_Φ is the set of all positive real elements of H .

We claim that Φ is full. Suppose first that $h = h_1$ is real. Then $h_1 = h_1^+ - h_1^-$ where $\langle h_1^+, e_i \rangle = \max\{0, \langle h_1, e_i \rangle\}$ and $\langle h_1^-, e_i \rangle = \max\{0, -\langle h_1, e_i \rangle\}$ for all i . Note that $\|h_1\|^2 = \|h_1^+\|^2 + \|h_1^-\|^2$. Let k_n^+ be the element of H_0 with components $\langle k_n^+, e_i \rangle$ given by

$$(5.6) \quad \langle k_n^+, e_i \rangle = \begin{cases} r_i & i \leq N \\ 0 & i > N \end{cases}$$

where N is chosen so that

$$(5.7) \quad \sum_{N+1}^{\infty} |\langle h_1^+, e_i \rangle|^2 < \frac{1}{2n^2} \|h_1^+\|^2$$

and r_i is a nonnegative rational such that

$$(5.8) \quad \langle h_1^+, e_i \rangle \geq r_i \|h_1^+\rangle > \left\{ \langle h_1^+, e_i \rangle - \frac{\|h_1^+\|}{n\sqrt{2N}} \right\}$$

Clearly $\|k_n^+ - \frac{h_1^+}{\|h_1^+\|}\| < 1/n$. It follows that $(h_1^+, k_n^+) \rightarrow \|h_1^+\|$ as $n \rightarrow \infty$.

Similarly, there is a sequence k_n^- in H_0 such that $(h_1^-, k_n^-) \rightarrow \|h_1^-\|$ as $n \rightarrow \infty$.

Noting that for any $h = h_1 + ih_2$ in H , $|(h, \varphi)| \geq \max\{|(h_1, \varphi)|, |(h_2, \varphi)|\}$, we have $\|h_1\|^2 = \|h_1^+\|^2 + \|h_1^-\|^2 = \lim_{n \rightarrow \infty} (h_1^+, k_n^+)^2 + \lim_{n \rightarrow \infty} (h_1^-, k_n^-)^2 \leq$

$\overline{\lim} |(h_1^+, k_n^+)|^2 + \overline{\lim} |(h_1^-, k_n^-)|^2 \leq 2 \sup_{\varphi \in \Phi} |(h_1, \varphi)|^2$. Now, if $h = h_1 + ih_2$ is

any element of H , then $\|h\|^2 = \|h_1\|^2 + \|h_2\|^2 \leq 2 \sup_{\varphi \in \Phi} |(h_1, \varphi)|^2 +$

$2 \sup_{\varphi \in \Phi} |(h_2, \varphi)|^2 \leq 4 \sup_{\varphi \in \Phi} |(h, \varphi)|^2$. Since $\|\varphi\| \leq 1$ if $\varphi \in \Phi = H_0$, we deduce

that $\|h\| \leq 2 \sup_{\substack{\varphi \in \Phi \\ \varphi \neq 0}} \{|(h, \varphi)| / \|\varphi\|\}$ for all h in H . Thus, Φ is full.

EXAMPLE 5.9. Let H be a separable Hilbert space and let $X = \mathcal{L}(H, H)$ be the space of bounded linear maps of H into itself. Let H_0 be a countable dense subset of the closed unit ball in H and let $\Phi = \{\varphi \in X^* :$

$(T, \varphi) = \langle Tk, k \rangle$, for some k in $H_0\}$. The cone K_Φ is the set of positive operators in $\mathcal{L}(H, H)$.

Since $\|T\| \leq 2 \sup_{\|h\| \leq 1} |\langle Th, h \rangle|$ for T in

$\mathcal{L}(H, H)$ and since $\|\varphi\| \leq \|k\|^2 \leq 1$ for k in H_0 , we have $\|T\| \leq$

$2 \sup_{\substack{\varphi \in \Phi \\ \varphi \neq 0}} \{|(T, \varphi)| / \|\varphi\|\}$. In other words, Φ is full.

EXAMPLE 5.10. Let \mathcal{D} be a bounded domain in R^n and let $X = L_p(\mathcal{D}, C)$

where $1 < p < \infty$. Let Σ be the Borel field of \mathcal{D} . Then Σ is a separable metric space with respect to the usual metric $d(E, E') = \mu(E \Delta E')$.

Let $\Sigma_0 = \{E_i\}$ be a countable dense set in Σ which include all hyper-

cubes with rational vertices contained in \mathcal{D} and let $Q = \{a+bi \in C :$

a, b rational $\}$. Let \mathcal{S} be the set of simple functions of the form

$\sum_{i=1}^n q_i \chi_{E_i}$ where the q_i are in Q and the E_i are disjoint elements of Σ_0 .

Note that \mathcal{S} is a countable subset of $L_q(\mathcal{D}, C)$ where $1/p + 1/q = 1$. It is easy to check that \mathcal{S} is dense in $L_q(\mathcal{D}, C)$. An element $\sum_{i=1}^n q_i \chi_{E_i}$ of \mathcal{S} is positive real if q_i is a nonnegative real number for $i = 1, \dots, n$.

Let Φ be the subset of \mathcal{S} consisting of all the positive real elements.

Since $X^* = L_p(\mathcal{D}, C)^* = L_q(\mathcal{D}, C)$, $\Phi \subset X^*$ and the cone K_Φ is simply the set of nonnegative functions in $L_p(\mathcal{D}, C)$. The proof that Φ is full is straightforward and is, therefore, left to the reader. Theorem 4.1, when interpreted in this context, becomes:

COROLLARY 5.11. If p is an element of $L_\infty(G, L_p(\mathcal{D}, C))$ such that $\int_G \int_G \xi(g) \bar{\xi}(g') p(g-g') d\mu d\mu$ is a nonnegative function in $L_p(\mathcal{D}, C)$ for all $\xi(\cdot)$ in $L_1(G, C)$, then $p(g) = \int_{\hat{G}} \gamma(g) d\nu$ where ν is a weakly regular measure on \hat{G} such that $\nu(F)$ is a nonnegative function in $L_p(\mathcal{D}, C)$ for all F in $\Sigma(\hat{G})$, and conversely.

This corollary plays a role in the study of positive solutions of certain partial differential equations.

EXAMPLE 5.12. Let H be a separable Hilbert space and let $\mathcal{L} = \mathcal{L}(H, H)$ be the closed ideal of compact operators in $\mathcal{L}(H, H)$. It is well-known ([8]) that $\mathcal{L}(H, H)^* = \mathcal{L}_1 \oplus \mathcal{L}^\perp$ where \mathcal{L}^\perp is the annihilator of \mathcal{L} and \mathcal{L}_1 is the trace class. Moreover, \mathcal{L}_1 is isometrically isomorphic with \mathcal{L}^* and \mathcal{L}^{**} is isometrically isomorphic with $\mathcal{L}_1^* = \mathcal{L}(H, H)$. Now let H_0 be a countable dense subset of the closed unit ball in H and let $\Phi = \{\varphi \in \mathcal{L}^*: (T, \varphi) = \langle Tk, k \rangle \text{ some } k \text{ in } H_0\}$. The cone K_Φ is the set of positive compact operators and Φ is a countable full family.

References

- [1] S. Bochner, Monotone funktionen, Stieltjessche integrale, und harmonische analyse, Math. Ann. 108, 378-410 (1933).
- [2] W. Rudin, "Fourier Analysis on Groups", Interscience, New York, 1962.
- [3] L. Loomis, "An Introduction to Abstract Harmonic Analysis", Van Nostrand, New York, 1953.
- [4] P. Falb, On a theorem of Bochner, Institut Haute Etudes Scientifique Publications Mathematiques, 36, (1969).
- [5] U. Haussmann, The inversion and Plancherel theorems in a Banach space, (to appear).
- [6] M. Naimark, "Normed Rings", P. Noordhoff, Groningen, 1959.
- [7] N. Dunford and J. Schwartz, "Linear Operators, Part I: General Theory", Interscience, New York, 1958.
- [8] J. Dixmier, "Les Algebres d'Operateurs dans l'Espace Hilbertien", Gauthiers-Villars, Paris, 1957.
- [9] N. Dinculeanu, "Vector Measures", Pergamon Press, New York, 1967.
- [10] P. Halmos, "Measure Theory", Van Nostrand, New York, 1950.
- [11] U. Haussmann, "Harmonic Analysis in Banach Space", Ph.D. Dissertation, June 1970, Brown University.